Correspondence

Linear Tapers in Rectangular Waveguides*

Recently, the special case of a linear double taper in rectangular waveguide propagating the TE_{10} mode in vacuum dielectric was examined.[1] Approximate expressions for the reflection coefficient and voltage standing-wave ratio as functions of the taper dimensions and free space wavelength were derived and experimentally verified. This correspondence generalizes the equations to be applicable for waveguides filled with dielectrics of arbitrary relative permittivity κ. As a matter of convenience, equations are numbered to correspond with similar equations in the referenced paper.

The approximate expression for the reflection coefficient is

$$\Gamma = \frac{1}{4\gamma_0}\left(\frac{d}{dx}\ln Z\right)_0 - \frac{1}{4\gamma_1}\left(\frac{d}{dx}\ln Z\right)_1 \cdot \exp\left[-2\int_0^L \gamma dx\right], \quad (6)$$

where γ is the propagation constant, Z is the characteristic impedance and L is the physical length of the taper. The subscripts 0 and 1 refer to conditions at the initial and terminal ends of the taper, respectively.

Consider a linear taper of length L connecting rectangular waveguides of impedance Z_0 and Z_1. In the taper section, the width and height of the guide are linear functions of the position:

$$a = a(x) = a_0 + \frac{a_1 - a_0}{L} x,$$

$$b = b(x) = b_0 + \frac{b_1 - b_0}{L} x.$$

To interpret (6) in terms of the TE_{10} mode in rectangular waveguide, the integrated characteristic impedance defined on a voltage-current basis is used. Let

$$Z = \frac{\pi\eta_0}{2} \frac{b}{a\sqrt{\kappa - (\lambda/2a)^2}} \quad (7)$$

and

$$\gamma = i\frac{2\pi}{\lambda_g} = i\frac{2\pi}{\lambda}\sqrt{\kappa - (\lambda/2a)^2}, \quad (8)$$

where η_0 is the impedance of free space, κ is the relative permittivity (dielectric constant) of the medium within the waveguide, λ is the free-space wavelength and λ_g is the guide wavelength. The logarithmic derivative in the taper is then found to be

$$\frac{d}{dx}\ln Z = \frac{1}{L}\left[\frac{b_1 - b_0}{b} - \frac{a_1 - a_0}{a}\left(\frac{\kappa}{\kappa - (\lambda/2a)^2}\right)\right]. \quad (9)$$

Substitution of (8) and (9) into (6) yields the following expression for the reflection coefficient:

$$\Gamma = \frac{i}{8\pi L/\lambda}\left[K_1 \exp(-i4\pi l) - K_0\right], \quad (11)$$

where

$$K_0 = \frac{\dfrac{b_1 - b_0}{b_0} - \dfrac{a_1 - a_0}{a_0}\left(\dfrac{\kappa}{\kappa - (\lambda/2a_0)^2}\right)}{[\kappa - (\lambda/2a_0)^2]^{1/2}} \quad (12)$$

$$K_1 = \frac{\dfrac{b_1 - b_0}{b_1} - \dfrac{a_1 - a_0}{a_1}\left(\dfrac{\kappa}{\kappa - (\lambda/2a_1)^2}\right)}{[\kappa - (\lambda/2a_1)^2]^{1/2}}, \quad (13)$$

and

$$l = \int_0^L \frac{dx}{\lambda_g} = \frac{1}{\lambda}\int_0^L \sqrt{\kappa - (\lambda/2a)^2}\, dx. \quad (14)$$

Eq. (14) may be integrated with the result that

$$l = \frac{L}{2(a_1 - a_0)}\left[\frac{2a_1}{\lambda_{g_1}} - \frac{2a_0}{\lambda_{g_0}} + \tan^{-1}\frac{2a_0}{\lambda_{g_0}} - \tan^{-1}\frac{2a_1}{\lambda_{g_1}}\right], \quad (15)$$

where

$$\lambda_{g_0} = \lambda/\sqrt{\kappa - (\lambda/2a_0)^2}$$

and

$$\lambda_{g_1} = \lambda/\sqrt{\kappa - (\lambda/2a_1)^2}.$$

The absolute magnitude of the reflection coefficient is

$$|\Gamma| = \frac{1}{L/\lambda}\left[\frac{K_0^2 + K_1^2}{64\pi^2} - \frac{K_0 K_1}{32\pi^2}\cos(4\pi l)\right]^{1/2}. \quad (16)$$

Using

$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|}, \quad (17)$$

the dominant mode voltage standing-wave ratio (VSWR) can be calculated as a function of frequency and taper length for a linear taper connecting two specified rectangular waveguides. The above equations reduce to those of the referenced paper if κ is replaced by unity.

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[1] R. C. Johnson, "Design of linear double tapers in rectangular waveguide," IRE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES, Vol. MTT-7, pp. 374–378; July, 1959. (Corrections: Vol. MTT-8, p. 458; July, 1960.)

The Unloaded Q of a YIG Resonator from X-Band to 4 Millimeters*

Carter and Flammer[1] have reported measurements at lower microwave frequencies of Q_u, the unloaded Q of a single crystal YIG sphere treated as a resonator. In their paper a comparison of the measured and theoretical Q_u, the latter computed on the basis of a constant relaxation time, yielded good agreement only in the 2–5 kMc range.

In this paper we report results of the variation of Q_u with frequencies from 9.5 to 67.8 kMc. Our measurements indicate that in this range Q_u is approximately constant.

Q_u may be found theoretically by solving the equation of motion of the freely processing magnetization in a saturated ferrimagnet. This equation, with Landau-Lifshitz damping, is

$$\dot{\boldsymbol{M}} = \gamma(\boldsymbol{M} \times \boldsymbol{H}) + \frac{\gamma_\alpha}{M_s}\boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}). \quad (1)$$

We consider the general ellipsoid with principal axes parallel to the x-y-z directions, and with the applied field H_0 in the z direction. Using the MKS rationalized system of units with $\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{M}$, the differential equation for m_x is

$$\ddot{m}_x + \alpha|\gamma|\left[2H_0 + [N_x + N_y - 2N_z]\frac{M_s}{\mu_0}\right]\dot{m}_x + \omega_0^2 m_x = 0 \quad (2)$$

where

$$\omega_0 = |\gamma|\left[\left(H_0 + (N_x - N_z)\frac{M_s}{\mu_0}\right)\cdot\left(H_0 + (N_y - N_z)\frac{M_s}{\mu_0}\right)\right]^{1/2}.$$

Q_u is therefore given by[2]

$$Q_u = \frac{\omega_0/|\gamma|}{\alpha\left[\dfrac{4\omega_0^2}{\gamma^2} + \left(\dfrac{M_s}{\mu_0}(N_x - N_y)\right)^2\right]^{1/2}}. \quad (3)$$

For a sphere, where $\omega_0 = \gamma H_0$,

$$Q_u = \frac{1}{2\alpha}. \quad (4)$$

To compute Q_u from the equation of motion with Bloch-Bloembergen damping,[3] we use

$$\dot{M}_{x,y} = \gamma(\boldsymbol{M} \times \boldsymbol{H})_{x,y} - \frac{M_{x,y}}{T_2}. \quad (5)$$

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[1] P. S. Carter and C. Flammer, "Unloaded Q of single crystal yttrium-iron garnet resonator as a function of frequency," IRE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-8, pp. 570–571; September, 1960.

[2] A. R. Von Hippel, "Dielectrics and Waves," J. Wiley and Sons, Inc., New York, N. Y., pp. 101–102; 1954.

[3] N. Bloembergen, "On the ferromagnetic resonance in nickel and supermalloy," Phys. Rev., vol. 78, pp. 572–580; June, 1950.

(At low signal levels, $\dot{M}_z \simeq 0$.) In this case,

$$Q_u = \frac{\omega_0 T_2}{2}. \quad (6)$$

The half-line width ΔH may be expressed as

$$\Delta H = \frac{\alpha\omega_0}{|\gamma|} = \frac{1}{|\gamma| T_2}. \quad (7)$$

If T_2 is a constant, ΔH remains constant and Q_u increases linearly with frequency. On the other hand, if α is constant, then Q_u remains constant, and ΔH increases linearly with frequency.

We performed measurements on a highly polished single crystal YIG sphere (0.020-inch diameter), mounted in a shorted section of waveguide. Using a modification of the method described by Lebowitz,[4] we determined the coupling coefficient β, and the loaded Q, Q_L, at the four frequencies shown in Table I. Q_u is then given by $Q_u = Q_L(1+\beta)$.

Commercial 4-mm wave meters were of insufficient resolution to determine Δf in the measurement of Q_L. Here precise frequency differences were determined by an interferometer technique.

TABLE I

f(kMc)	Q_u	α	ΔH(oe)
9.48	3.3×10^3	1.5×10^{-4}	0.5
17.18	2.9×10^3	1.7×10^{-4}	1.0
35.2	3.1×10^3	1.6×10^{-4}	2.0
67.8	2.3×10^3	$2.2\times10^{-}$	5.4

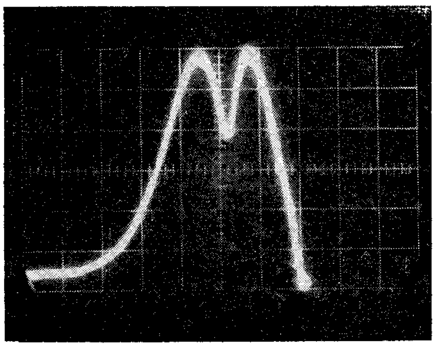

Fig. 1.

The results of Table I show that over this wide range Q_u and α are nearly independent of frequency; the assumption of a constant ΔH and T_2 is, therefore, not borne out by experiment.

It is interesting to note that the material maintains a reasonably high value of Q_u well into the mm wave range. Fig. 1 illustrates the response at the detuned open, while the klystron is swept over a mode in the 4-mm range. Thus, the use of highly polished YIG spheres as resonators in filter circuits[5] may be extended well into the mm wave range.

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[4] R. A. Lebowitz, "Determination of the parameters of cavities terminating transmission lines," IRE Transactions on Microwave Theory and Techniques, vol. MTT-4, pp. 51–53; January, 1956.

[5] P. S. Carter, Jr., "Magnetically tunable microwave filters employing single crystal garnet resonators," 1960 IRE International Convention Record, pt. 3, pp. 130–135.

Z_0 of Rectangular Coax*

The characteristic impedance Z_0 of rectangular coax has long been the subject of experimental and theoretical investigations. The problem is to obtain an expression for Z_0 which is both simple and precise to facilitate device design. The recent works of Chen[1] and Cohn[2] are summarized in Fig. 1 and compared with simple and precise calculations already known.[3,4]

The curve for $b/g \geq 1$ comes from Chen's equation (3) plus equation (4). The curve $b/g=0$ comes from Cohn's[5] equation (15). The intermediate curves for $g/h \leq 1$ come from scaling Cohn's[2] Fig. 4 to connect $b/g=0$ and $b/g \geq 1$. The intermediate curves for $g/h > 1$ are based on Chen's approximation.[6]

Cohn's Fig. 4 is most accurate below $g/h = 0.25$. An accurate plot for intermediate values of b/g is known for $g/h = 1$[7] and is shown as curve C in Fig. 2(a). In Fig. 2,

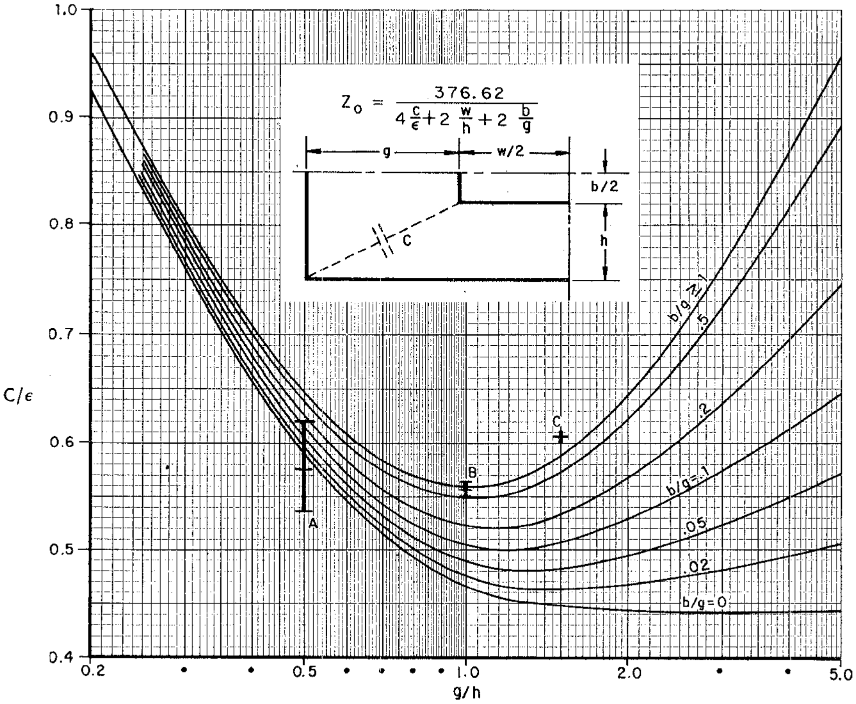
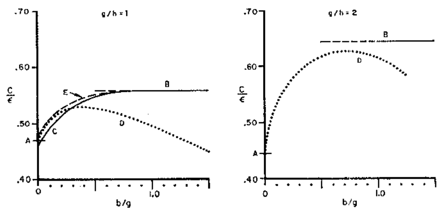


Fig. 1—Capacitance of one corner of rectangular coax.

The principal factor determining the Z_0 of TEM transmission line is its capacitance per unit length. Since the capacitance between parallel plates is readily calculated, the problem of calculating Z_0 of rectangular coax reduces to that of determining its "corner" capacitance. Assuming $w/h \geq 1$, Fig. 1 gives the corner capacitance. The characteristic impedance of air-filled rectangular coax line may then be obtained from

$$Z_0 = \frac{376.62}{4\frac{C}{\epsilon} + 2\frac{w}{h} + 2\frac{b}{g}}.$$

For $b/g > 1$, this equation and Fig. 1 may also be used for eccentric lines; however, each capacitive term will appear separately in the denominator.



(a) (b)

Fig. 2—Corner capacitance or intermediate values of b/g.

points A are from the theoretically derived curve for $b/g=0$; lines B are from the theoretically derived curve for $b/g \geq 1$. The validity of the assumption that Cohn's Fig. 4 can be stretched and made to apply is demonstrated in Fig. 2(a). Curve E is based on this assumption, and it is seen that if curve C were compressed to intersect point A and line B, the difference between the curves would be negligible.

Curves D are based on Chen's approximation. The validity of Chen's approximation is demonstrated by the closeness of

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[1] T. S. Chen, "Determination of the capacitance, inductance, and characteristic impedance of rectangular lines," IRE Transactions on Microwave Theory and Techniques, vol. MTT-8, pp. 510–519; September, 1960.

[2] S. B. Cohn, "Thickness corrections for capacitive obstacles and strip conductors," IRE Transactions on Microwave Theory and Techniques, vol. MTT-8, pp. 638–644; November, 1960.

[3] Y. A. Omar and C. F. Miller, "Characteristic impedance of rectangular coaxial transmission lines," *Trans. AIEE*, vol. 71, pp. 81–89; January, 1952.

[4] J. J. Skiles and T. J. Higgins, "Determination of the characteristic impedance of UHF coaxial rectangular transmission lines," *Proc. Natl. Electronics Conf.*, Chicago, Ill., October 4–6, 1954, vol. 10, pp. 97–108; 1954.

[5] S. B. Cohn, "Shielded coupled-strip transmission line," IRE Transactions on Microwave Theory and Techniques, vol. MTT-3, pp. 29–38; October, 1955.

[6] Chen, *op. cit.*, equation (26).

[7] *Ibid.*, Fig. 8.